Path transformations in Lévy processes

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1. Introduction

Let \( D \) be the space of c.à.d. l.à.g. functions with finite or infinite life time. We are interested in maps from \( D \) onto itself which connect the paths of stochastic processes related to Lévy processes. The existence of such path transformations is mainly due to the special properties of the increments of Lévy processes and their main interest is to provide and explain some identities in law between remarkable functionals. In most of the cases such identities are first noticed by pure calculation in particular cases like Brownian motion or Cauchy processes. Then, it sometimes arises that they have pathwise explanation which can be spread to a large class of Lévy processes. Moreover, these path transformations usually lead to more complete identities in law than the initial one.

To illustrate this topic, we chose to present two examples. The first one is known as Ver-vaat’s transformation and its aim was to explain the identity in law between the amplitude of the Brownian bridge and the maximum of the normalized Brownian excursion. The second one is due to Bertoin, (1993). It provides a pathwise explanation of the identity in law between the time spent under 0 by a Lévy process up to time 1 and the time at which this process first hits its minimum before 1.

2. Normalized excursion and bridge for stable processes

Let \( X \) be a spectrally positive stable process, starting from 0, with index \( \alpha \in (0, 2] \). That is a Lévy process with no negative jumps which satisfies the following scaling property:

\[
(X_t)_{t \geq 0} \overset{d}{=} (s^{-1} X_{s^\alpha t})_{t \geq 0},
\]

for every \( s > 0 \). Denote by \( b \) the bridge of the process \( X \) with length 1. It is defined to be the process with lifetime 1 and with the law of \( X \) conditioned to return to 0 at time 1:

\[
(b_t, 0 \mathbin{\#} t \mathbin{\#} 1) \overset{d}{=} (X_t, 0 \mathbin{\#} t \mathbin{\#} 1)_{t < 1} | X_1 = 0.
\]

When \( \alpha \in (1, 2] \), 0 is not polar. This property allows us to construct the paths of \( b \) from those of \( X \) by the following way: Let \( t > 1 \) and \( g_t \) be the last time before \( t \) at which \( X \) reaches 0, \( g_t = \sup \{ s \mathbin{\#} t : X_s = 0 \} \). Then the process,

\[
(g_t^{-1/\alpha} X_{s \cdot g_t}, 0 \mathbin{\#} s \mathbin{\#} 1),
\]

is a bridge.

Now, denote by \( e \) the normalized excursion of \( X \), that is the process with lifetime 1 and with the law of \( X \) conditioned to stay positive on the time interval \((0, 1)\) and to return to 0 at time 1:

\[
(e_t, 0 \mathbin{\#} t \mathbin{\#} 1) \overset{d}{=} (X_t, 0 \mathbin{\#} t \mathbin{\#} 1)_{X_t > 0, 0 < t < 1, X_1 = 0}.
\]
Like for $b$, there exists a pathwise construction of $e$ from the path of $X$. Denote by $X_t$ the minimum of $X$ over $[0,t]$, i.e.: $X_t = \inf_{u \leq t} X_u$. For $t > 1$, let $\underline{g}_t$ and $\underline{d}_t$ be respectively the last time before $t$ and the first time after $t$ at which $X$ reaches his absolute minimum over $[0,t]$:

$$g_t = \sup \{ s \cap t : X_s = X_t \}, \quad d_t = \inf \{ s \cap t : X_s = X_t \}.$$ 

Then the process,

$$([\underline{d}_t \cap \underline{g}_t])^{-1/\alpha} (X_t - X_s)_{\underline{g}_t + ([\underline{d}_t - \underline{g}_t])s}, 0 \cap s \cap 1,$$

is a normalized excursion. Note that this construction is valid for any $\alpha \in (0,2]$.

In Theorem 1, we present a relationship between the paths of the bridge and those of the normalized excursion. In the Brownian case, this transformation is due to Vervaat, (1979), see also Bertoin and Pitman, (1994), and was extended by Chaumont, (1997a) to stable Levy processes. Let $m$ be the first time at which the bridge $b$ hits its minimum over $[0,1]$, that is $m = \inf \{ t : b_t = \sup_{s \leq t} b_s \}$. For every path $\omega \in \mathcal{D}$ with lifetime 1 and every random time $T \in [0,1]$, denote also by $V[\omega, T]$ the path transformation which consists in splitting the path $\omega$ at time $T$ and then in inverting the two parts so obtained, that is:

$$V[\omega, T] := (\omega_{T + u \pmod 1} \cap \omega_T, 0 \cap u \cap 1).$$

Then for every spectrally positive stable Levy process, the following path transformation holds:

**Theorem 1** $m$ is almost surely unique and the process $V[b, m]$ is a normalized excursion.

Note that the transformation $\omega \mapsto V[\omega, T]$ preserves the amplitude, $\sup_{0 \leq t \leq 1} \omega_t \cap \inf_{0 \leq t \leq 1} \omega_t$, of the process $\omega$, so an important consequence of the previous result is that the amplitude of the bridge is distributed as the absolute maximum of the normalized excursion. Another special property verified by the normalized excursion $V[b, m]$ is that it is independent of the time spent by $b$ under $0$: $\int_0^t \mathbb{1}_{\{b_u \leq 0\}} du$. Although this property can not be seen as a direct consequence of the above theorem, it can be drawn from the proof given by Chaumont, (1997b), see also Biane, (1986).

In the Brownian case, it was noticed by Biane, (1986), that such a transformation is not reversible. Indeed, the time $m$ is independent of the process $V[b, m]$. However, by introducing an independent random variable with the same distribution as $m$, it is possible to construct a Brownian bridge from a normalized Brownian excursion. In the more general case of spectrally positive stable Levy processes, the same situation still holds.

**Theorem 2** Let $U$ be a uniformly distributed random value over $[0,1]$, independent of the normalized excursion $e$.

Then the process $b' := V[e, U]$ is a Brownian bridge and $1 \cup l = \inf \{ t \cup 0 : b'_t = \sup_{0 \leq u \leq 1} b'_u \}$.

As a consequence of this theorem, we get that the time when the bridge reaches its minimum is uniformly distributed.

The proof of the above theorems essentially uses the ‘exchangeability’ of the increments of $X$. Indeed, it can be proved that it still holds for every stochastic process with exchangeable increments which return to 0 at time 1 and such that the zero set has zero Lebesgue measure. More generally, let $Y$ be such a process, let also $m_t$ be the first hitting time of the minimum before $t$, $(t < 1)$ by $Y$ and $m_T$ be the last hitting time of the maximum before $1 \cup t$ by the process $(Y_t \cup s \cap Y_t, 0 \cup s \cup 1 \cup t)$, then the sum

$$m_t + m_T$$

is uniformly distributed over $[0,1]$. 
This identity was noticed by Knight, (1999) for $t = 0$ and Chaumont, (1999) in the general case. The path transformation explaining this result can be found in Chaumont, (1997b).

3. Splitting at the infimum of Lévy processes

In the present section, $X$ is any Lévy process. Fix $t > 0$ and let $m_t$ be the last time at which $X$ hits its minimum before time $t$: $m_t = \sup \{ s : t \cap X_s = \inf_{u \leq t} X_u \}$. We are going to present a pathwise transformation which leads to the following identity in law:

$$ (1) \quad \int_0^t \mathbb{1}_{\{X_u < 0\}} \, du \overset{(d)}{=} m_t. $$

In the Brownian case, this relation was first noticed by Paul Lévy, (1939) who proved that each member is arcsine distributed. Then, Sparre-Andersen, (1953) and Spitzer, (1956) gave an interpretation of this identity in discrete time, that is for random walks. Here, we give the path construction of Bertoin, (1993) for Lévy processes which involves two pairs of processes that we first introduce.

Let us define $\bar{X}$ be the post-minimum process of $(X_s, 0 \sqcap s \sqsupseteq t)$:

$$ X_s = X_{m_t+s} \sqcap X_{m_t}, \quad 0 \sqcap s \sqcap t \sqqueteq m_t, $$

and $X$ be its reversed pre-minimum process:

$$ X_s = X_{(m_t-s)-} \sqcap X_{m_t}, \quad 0 \sqcap s \sqqueteq m_t. $$

Now let $L$ be the (semi-martingale) local time at level 0 of $X$. $L$ is the continuous increasing process which is specified by the Meyer-Tanaka formula. For every $s \sqcap t$, we also introduce

$$ A^+_s = \int_0^s \mathbb{1}_{\{X_u > 0\}} \, du \quad \text{and} \quad A^-_s = \int_0^s \mathbb{1}_{\{X_u \leq 0\}} \, du, $$

the times spent by $X$ respectively in $(0, \infty)$ and in $[-\infty, 0]$, and their right-continuous inverses $\alpha^+_s = \inf \{ u : A^+_u > s \}$. It remains to define the processes $X^\dagger$ and $X_{\dagger}$ as follows:

$$ X^\dagger_s = \left( X \sqcap \sum_{0 < s \leq t} (\mathbb{1}_{\{X_u \leq 0\}} X^+_s + \mathbb{1}_{\{X_u > 0\}} X^-_s) + \frac{1}{2} L \right) (\alpha^+_s), \quad s \sqqueteq A^+_t. $$

To imagine how $X^\dagger$ is defined, note that when $X$ has no Gaussian component, $L \sqqueteq 0$ and then, $X^\dagger$ is obtained from $X$ after the juxtaposition of its excursions in $(0, \infty)$. Similarly, we define

$$ X_{\dagger} = \left( X \sqcap \sum_{0 < s \leq t} (\mathbb{1}_{\{X_u \leq 0\}} X^+_s + \mathbb{1}_{\{X_u > 0\}} X^-_s) \sqqueteq \frac{1}{2} L \right) (\alpha^-_s), \quad s \sqqueteq A^-_t. $$

Here is Bertoin’s result:

**Theorem 3 (Bertoin, 93)** For every $t > 0$, the pairs of processes $(X, \sqcap X)$ and $(X^\dagger, X_{\dagger})$ have the same law.

Observe that the lifetimes of $X$ and $X^\dagger$ are respectively $m_t$ and $A^-_t$, so the above theorem implies identity (1). Moreover, one easily check that

$$ X_t = X^\dagger_{A^-_t} + X_{A^-_t} = X_{t-m_t} \sqcap X_{m_t}, $$
which means that identity (1) holds conditionally on $X_t$. In other words we get the following more complete identity in law

$$(A_t^{-}, X_t) \overset{d}{=} (m_t, X_t).$$

REFERENCES


FRENCH RÉSUMÉ

Cet exposé a pour but de présenter l’intérêt des transformations trajectorielles liées aux processus de Lévy. Deux exemples illustrent ce sujet : la transformation de Vervaat entre le pont et l’excursion normalisée et une transformation due à Bertoin qui entraîne l’identité entre la loi du temps passé au dessous de 0, par un processus de Lévy, avant le temps 1 et celle du temps où ce processus atteint son minimum absolu, sur l’intervalle $[0, 1]$. 