On the Number Of Equilibrium States in Weakly Coupled Random Networks

Akira Date\(^1\)
Chii-Ruey Hwang\(^2\)
Shuenn-Jyi Sheu\(^2\)
\(^1\)Department of System and Human Science, Osaka University, Osaka, Japan
\(^2\)Institute of Mathematics, Academia Sinica, Taipei, Taiwan. crhwang@sinica.edu.tw

The number of equilibrium states (fixed points) is an important issue in the study of the dynamics of neural networks and statistical physics. The equilibrium states correspond to stored patterns (memory). The point of the dynamics is to recover a stored pattern when given a distorted pattern as an initial condition. In other words, the initial network state follows a neural dynamical trajectory and arrives at an equilibrium state. Roughly speaking, the number of equilibrium states corresponds to the size of the memory.

We study the asymptotic behavior of the expected number of equilibrium states for some Hopfield-like model. In this model the connection weights (connectivities) consist of symmetric and antisymmetric parts. Note that the methods used in treating the symmetric case cannot be generalized to asymmetric connectivities with a larger antisymmetric part, i.e., \(c \geq 1\) in (1.1) below. Asymmetry is essential when learning is taken into consideration. However, the asymptotic behaviors become quite difficult to analyze.

Now consider a fully interconnected network consisting of \(n\) elements (neurons) in which the \(i\)th output \(x_i, i = 1, \cdots, n\), takes values 1 or -1. \(w_{ij}, 1 \leq i, j \leq n\), the connection weight from the \(j\)th element to the \(i\)th element is a composition of two components, namely,

\[
w_{ij} = w^s_{ij} + cw^a_{ij},
\]

where \(\{w^s_{ij}, w^a_{ij} : i \leq j\}\) are i. i. d. \(\mathcal{N}(0,1)\), \(w^s_{ij} = w^s_{ji}, w^a_{ij} = -w^a_{ji}\), \(c\) is a constant.

\(w^s_{ij}\) and \(w^a_{ij}\) represent the symmetric weight and the antisymmetric weight respectively. \(c\) denotes the relative weight of these two components; hence, it suffices to consider \(c \geq 0\). One may regard \(c = \infty\) as a completely antisymmetric case, i. e. \(w_{ij} = w^a_{ij}\).

Let \(T\) denote the (random) nonlinear transformation from \(\{-1,1\}^n\) to \(\{-1,1\}^n\) defined by

\[
Tx = \{\text{sgn}(\sum_{j=1}^{n} w_{ij}x_j)\}_{i=1,\cdots,n},
\]

where \(\text{sgn}(u)=1\) when \(u > 0\) and is -1 otherwise. And the dynamics of the network is described by \(x^{k+1} = Tx^k\), where \(x^k\) denotes the output at time \(k\). This dynamics is widely used as a first approximation of neural dynamics.

A point \(x\) in \(\{-1,1\}^n\) is called an equilibrium state (fixed point) if \(Tx = x\). We are interested in the asymptotics of the expected number of equilibrium states when \(n \to \infty\).

The approach here is applicable for all \(c\). The basic idea consists of using specific representations of Gaussian random vectors, conditioning on the minimum of i. i. d. Gaussian random variables, and large deviations.
RÉSUMÉ

Dans cet article, nous considérons un réseau complètement connexe qui consiste de \( n \) éléments de outputs \( x = \{x_i, x_i = +1, -1, 1 \leq i \leq n\} \), avec des poids \( w_{ij} = w_{ij}^s + cw_{ij}^a \) qui se décomposent d’une partie symétrique et une partie anti-symétrique, et une dynamique décrite par \( x \mapsto \{\text{sign}(\sum w_{ij}x_j)\} \). Ici, \( \{w_{ij}^s, w_{ij}^a, w_{kk}, i < j\} \) sont des variables aléatoires indépendantes et identiquement distribuées de loi \( \mathcal{N}(0, 1) \), où \( w_{ij}^s = w_{ji}^s \), \( -w_{ij}^a = w_{ji}^a \), et \( c \) est une constante à valeur dans \([0, \infty]\). Le cas où \( c = \infty \) signifie que \( w_{ij} = w_{ij}^a \). Le comportement asymptotique du nombre moyen des états d’équilibre du réseau est aussi étudié.