NUMERICAL EVALUATION OF MULTIVARIATE INTEGRALS
OVER ELLIPSOIDAL REGIONS

Paul N. Somerville
University of Central Florida
Orlando, FL 32903
e-mail somervil@pegasus.cc.ucf.edu

1. INTRODUCTION

Let \( x = (x_1, x_2, \ldots, x_k) \) have the distribution \( f(x) = \text{MVN}(\mu, \Sigma \sigma^2) \), where \( \Sigma \) is a known positive definite matrix and \( \sigma^2 \) is a constant. We wish to evaluate \( f(x) \) over \( \Lambda \), an ellipsoidal region given by \((x - \mu)^T \Sigma (x - \mu) = 1\). That is, we wish to obtain

\[
P = \int_{\Lambda} f(x) \, dx.
\]

The problem occurs in some reliability situations, and a special case is the evaluation of noncentral F and noncentral chi-square distributions.

If \( \sigma^2 \) is known, then without loss of generality, set \( \mu = 0, \sigma = 1 \), and let \( \Sigma \) be the correlation matrix. If \( \sigma^2 \) is estimated by \( s^2 \) with \( v \) degrees of freedom such that \( v s^2 / \sigma^2 \) is a chi-square variate, then, without loss of generality we may assume \( f(x) \) has the central multivariate-t distribution and \( \Sigma \) is the correlation matrix.

Somerville (1998a, 1998b, 1999) developed procedures for calculation of the multivariate normal and multivariate-t distributions over convex regions bounded by hyperplanes and over ellipsoidal regions and has written a Fortran 90 programs for the former. A Fortran 90 program MVELPS has been written to evaluate the integrals over ellipsoidal regions.

2. EVALUATION OF MULTIVARIATE NORMAL AND T INTEGRALS

Let \( \Sigma = TT^T \) (Cholesky decomposition) and set \( x = T w \). The random variables \( w_1, w_2, \ldots, w_k \) are independent standard normal or spherically symmetric t variables. Without loss of generality we may assume that the axes of the ellipsoid are parallel to the coordinate axes and the ellipsoid has the equation \((w - \mu)^T B^{-1} (w - \mu) = 1\), where \( B \) is a diagonal matrix with the \( i \)th diagonal element given by \( b_i \).

Case a) \( \sigma^2 \) is known (Multivariate Normal)

Let \( r^2 = w^T w \). Since \( \sigma^2 \) is known, \( r^2 \) is distributed as chi-square with \( k \) degrees of freedom. A random direction in the \( w \) coordinate system is selected. (It is convenient to use both the random direction and its negative). If the origin is in the ellipsoid, and the distance to the boundary is \( r(i) \), then an unbiased estimate of \( P \), (call it \( P(\sigma) \)), is \( Pr[r < |r(i)|] = Pr[\chi^2 < (r(i))^2] \). The procedure is repeated (say mocar times) and the resulting estimates averaged to obtain a preliminary estimate. The procedure may be repeated (say irep times) and the repetitions used to obtain a standard error.

If the origin is not in the ellipsoid, then if the line through the origin in the random direction does not intersect the ellipsoid, the estimate \( P(\sigma) \) is zero. If the line intersects the ellipsoid, then an estimate is

\[
P(\sigma) = Pr[r < |r(i)|] - Pr[r < |r^*(i)|] = Pr[\chi^2 < (r(i))^2] - Pr[\chi^2 < (r^*(i))^2]
\]

where \( |r(i)| > |r^*(i)| \) are the two distances from the origin to the ellipsoid boundary.

Case b) \( \sigma^2 \) is not known (Multivariate-t)

The ellipsoid is given by \((w - \mu)^T B^{-1} (w - \mu) = 1\). Suppose \( s = \sigma \), then the probability content of the ellipsoid would be \( P(\sigma) \) (say) as given in the previous section. Let \( g(s) \) be the frequency function for \( s \) (\( vs^2 / \sigma^2 \) has the \( \chi^2 \) distribution with \( v \) degrees of freedom). Then the unconditional probability content of the ellipsoid is given by

\[
P = \int_0^\infty P(s) g(s) \, ds.
\]

Using Gauss-Laguerre quadrature, we obtain an approximation

\[
P = \sum w_i g(s_i) P(s_i)
\]

where \( s_i \) and \( w_i \) are the quadrature abscissae and weights respectively. We use the formulae for \( P(\sigma) \) given for case a)
3. ACCURACIES AND COMPUTATION TIMES

When the ellipsoidal region is a hyperspheroid, MVELPS give probabilities for the non-central F or the non-central chi-square distributions. The results of MVELPS.FOR were compared with noncentral F and chi-square computer programs and in no case was there reason to suspect bias in MVELPS.FOR. Further, the standard error of estimate obtained was consistent with results of the noncentral programs. Using a PC with a Pentium 90 processor, running times were in seconds.

4. COMMENTS

The Fortran 90 program also computes probabilities using a “binning” method (Somerville 1998a,b). The binning procedure is always faster for the multivariate normal case, and is also faster, for a sufficiently large number of random directions, for multivariate t. It needs to be emphasized that, in addition to the number of random directions, the standard error of the estimate is a function of the shape and position of the ellipsoid (after the “standardizing” transformations). If the ellipsoid is a spheroid centered at the origin, the accuracy is independent of the number of random directions and is a function of the processor and the computer program. The standard error, in general, is a function of the degree of departure of the ellipsoid from a spheroid centered at the origin.

The program MVELPS has a shortcut method for the calculation of probabilities for the noncentral chi-square and F distributions. The shortcut method takes advantage of the fact that the regions are spheroids and that the probabilities are a function only of the radius and distance of the center from the origin. Thus running times and standard errors are decreased. For all cases examined, $10^4$ random directions resulted in standard errors of no more than .0004 and usually much less, with running times (Pentium 90 processor) of 2 or 3 seconds.

The program MVELPS can also be used to calculate probabilities for a function which is a linear combination of $\chi^2$ or, in cases where the F distributions have the same denominator degrees of freedom, F probabilities, provided the coefficients of $\chi^2$ or F are positive. This is because the probability may be interpreted as the content of an ellipsoid in a space which has dimension equal to the sum of the (numerator) degrees of freedom of the $\chi^2$ or F distributions.

5. SUMMARY AND CONCLUSIONS

A methodology has been developed and a Fortran 90 program written to find the probability content of an hyperellipsoid when the underlying distribution is multivariate normal or multivariate-t. Special cases are the noncentral F and $\chi^2$ distributions. The methodology also extends to the case where probabilities are required for linear combinations of $\chi^2$ or F probabilities provided the coefficients are positive.

REFERENCES


RESUME

Une methodologie est presente pour le calculation des integrales multivariate, normale et t, quand le region de calcule est une ellipse avec dimension k. Une program Fortran MVELPS pour calcule les integrales existe.