SOME CHARACTERISTIC PROPERTIES OF RECORD VALUES

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ABSTRACT

In this paper the basic concepts and properties of the record Values of univariate continuous distributions are presented. Some of their characteristic properties are discussed.

1.0 Introduction

Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function $F(x)$. Set $Y_n = \max (\min \{X_1, \ldots, X_n\})$, $n \geq 1$. We say $X_j$ is an upper (lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > \langle \langle \rangle Y_{j-1}, j \geq 1$. By definition, $X_1$ is an upper as well as a lower record value. Thus the upper record values in the sequence $\{X_n \geq 1\}$ are the successive maxima. One can go from lower records to upper records by replacing the original sequence of rv's by $\{X_j, j \geq 1\}$ or if $P(X_j > 0) = 1$ by $\{1/X_i, i \geq 1\}$. Unless mentioned otherwise we will call the upper record values as record values. The indices at which the record values occur are given by the record times $\{U(n)\}$, $n \geq 0$, where $U(n) = \min \{j|j > U(n-1), X_j > X_{U(n-1)} \} \min \{j|j > U(n-1), X_j > X_{U(n-1)} \}$ and $U(1) = 1$. The record times of the sequence $\{X_n, n \geq 1\}$ are the same as those for the sequence $\{F(X_n) \geq 1\}$. Since $F(X)$ has an uniform distribution, it follows that the distribution of $U(n)$, $n \geq 1$ does not depend on $F$. Properties of record values of i.i.d. rvs have been extensively studied in literature, for example, see Ahsanullah (1995). Many properties of the record value sequence can be expressed in terms of the function $R(x)$, where $R(x) = -\ln F(x)$, $0 < F(x) < 1$ and $F(x) = 1 - F(x)$. Here 'ln' is used for the natural logarithm.

The joint pdf $f(x_1, x_2, \ldots, x_n)$ of the n record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ is given by

$$f(x_1, x_2, \ldots, x_n) = r(x_1) r(x_2) \ldots r(x_{n-1}) f(x_n)$$

for $-\infty < x_1 < x_2 < \ldots < x_{n-1} < x_n < \infty$,

where $r(x) = \frac{d}{dx} R(x) = \frac{f(x)}{1-F(x)}$, $0 < F(x) < 1$. The function $r(x)$ is known as hazard rate. The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$f(x_i, x_j) = \frac{(R(x_j))^{i-1}}{(i-1)!} r(x_i) \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j)$$

for $-\infty < x_i < x_j < \infty$.

If we define $F_n(x)$ as the distribution function of $X_{U(n)}$ for $n \geq 1$, then we have

$$F_n(x) = P(X_{U(n)} \leq x) = \int_x ^ \infty \frac{R^{n-1}(u)}{(n-1)!} dF(u), \quad -\infty < x < \infty.$$}

The pdf $f_n(x)$ of $X_{U(n)}$ is

$$f_n(x) = \frac{R^{n-1}(x)}{(n-1)!} f(x), \quad -\infty < x < \infty.$$}

The marginal pdf of the nth lower record value as well as the joint pdf of $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ can be obtained similarly.
2. Main Results

Moments of Record Values
Let $M_{m(t)}$ be the moment generating function of $X_{n(t)}$, then it can be shown that

\begin{equation}
M_{m-1}(t) - M_{m(t)} = \frac{t}{\Gamma(n+1)} \int_{-\infty}^{\infty} e^{tx} \{ -\ln \bar{F}(x) \}^n \bar{F}(x) \, dx
\end{equation}

Let $M_{m,m}(t_1,t_2)$ be the joint moment generating function of $X_{n(m)}$ and $X_{n(t)}$ for $m < n$, then we have

\begin{equation}
M_{m,m+1}(t_1,t_2) - M_{m,m}(t_1,t_2) = \frac{t_2}{\Gamma(m)\Gamma(n-m+1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1x + t_2y} \{ -\ln \bar{F}(x) \}^{m-1} \{ -\ln \bar{F}(y) + \ln \bar{F}(x) \}^{n-m} \frac{f(x)}{\bar{F}(x)} \bar{F}(y) \, dy \, dx
\end{equation}

Using (2.1) and (2.2), we can obtain the recurrence moments of various distributions.

Examples
For the exponential distribution $F(x) = e^{-x}$, $x > 0$, we have

\begin{equation}
(1-t) M_{n+1}(t) = M_n(t), \quad t \neq 1
\end{equation}

and

\begin{equation}
(1-t) M_{m,n+1}(t_1,t_2) = M_{m,n}(t_1,t_2), \quad t_2 \neq 1
\end{equation}

Using the equations (2.3) and (2.4), we can show that for $n \geq 1$ and $r = 0, 1, 2, \ldots$

\begin{equation}
E(\bar{X}_{n(n)}^{r+1}) = E(\bar{X}_{n(n-1)}^{r+1}) + (r+1)E(\bar{X}_{n(n)}^r)
\end{equation}

and consequently, for $0 \leq m \leq n-1$ we can write

\begin{equation}
E(\bar{X}_{n(n)}^{r+1}) = E(\bar{X}_{n(n)}^{r+1}) + (r+1) \sum_{p=m+1}^{n} E(\bar{X}_{n(n)}^r)
\end{equation}

Similarly we can obtain

\begin{equation}
E(X_{n(n)}^r X_{n(n+1)}^p) = E(X_{n(n)}^r X_{n(n)}^p) + pE(X_{n(n)}^r X_{n(n+1)}^{p-1}), m, n=1,2,\ldots,m < n.
\end{equation}

For the logistic distribution $F(x) = \frac{1}{1+e^{-x}}$, $-\infty < x < \infty$, we have

\begin{equation}
(1-t) M_{n+1}(t) = M_n(t) + t M_{n+1}(t-1), \quad t \neq 1
\end{equation}

and

\begin{equation}
(1-t_2) M_{m,n+1}(t_1,t_2) = M_{m,n}(t_1,t_2) + t_2 M_{m,n+1}(t_1,t_2-1), \quad t_2 \neq 1
\end{equation}

Using the relations (2.8) and (2.9), we obtain

$E(\bar{X}_{n(n)}) = E(\bar{X}_{n(n-1)}) = \zeta(n)$, where $\zeta(n)$ is the Riemann zeta function of order $n$.

Similarly for the joint moment, we obtain

\begin{equation}
E(X_{n(n)} X_{n(n+1)}^r) = nE(X_{n(n)}^r) + mE(X_{n(n)}^r) + m\zeta(m+1) + m\zeta(n+1) - mn+1)
\end{equation}

REFERENCES


RÉSUMÉ

Dans ce document les concepts et propriétés de base de les records de valeurs de la distribution continue univariées sont présenté. Certaines relations entre des fonctions générer et des relations répétantes sont discuter