An Exact Iterated Bootstrap Algorithm For Small-Sample Bias Reduction

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1. Introduction

By considering bootstrap iteration as a Markov process, we propose an exact algorithm for its implementation in the context of small-sample bias reduction. The algorithm caters for any number of bootstrap iterations without the need for extensive Monte Carlo resampling , in the context of bootstrap bias reduction for small samples. The performance of high-level bootstrap iteration is then investigated.

2. Exact computation of bias-corrected estimator based on j bootstrap iterations

Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a random sample drawn from an unknown distribution F_0, and F_1 be the empirical distribution of $\mathcal{X}$. Let $\theta = t(F_0)$ be a functional of F_0. Then a natural estimator of θ is $\hat{\theta} = t(F_1)$, provided t is defined on the space of all discrete distributions. Hall (1992) gives a general formula for the bias-corrected estimator of θ based on j bootstrap iterations, namely

$$\hat{\theta}_j = \sum_{i=1}^{j+1} \binom{j+1}{i} (-1)^{i+1} \mathbb{E}\{t(F_i) \mid F_1\},$$

for $j = 1, 2, \ldots$, where F_i denotes the empirical distribution of a random sample of size n drawn from F_{i-1}.

We see that bias reduction by j bootstrap iterations involves calculation of the conditional expectations $\mathbb{E}\{t(F_i)|F_1\}$. Approximation to the latter is usually obtained by Monte Carlo simulation of nested bootstrap resamples, which becomes impractical if j is large. Following the notation of Fisher and Hall (1991), define $\mathcal{L}(n)$ to be the set of all distinct n-tuples $(l_1, \ldots, l_n)$ such that the l_i's are nonnegative integers satisfying $l_1 \geq l_2 \geq \cdots \geq l_n$ and $\sum_{i=1}^n l_i = n$. For each $l \in \mathcal{L}(n)$, define $\mathcal{M}(l)$ to be the set of all distinct permutations of l. Note that each, possibly iterated, bootstrap resample arising from $\mathcal{X}$ can be uniquely identified with an n-tuple $(k_1, \ldots, k_n)$ in $\mathcal{M}(l)$ for a unique $l \in \mathcal{L}(n)$, where k_i gives the number of appearances of X_i in the bootstrap resample. A jth level bootstrap resample relates stochastically to $\mathcal{X}$ by the j-step transition probabilities, which are obtainable as components of the product matrix $P_n^j = [p_n^{(j)}(s,t) : s, t \in \mathcal{L}(n)]$. For each $k = (k_1, \ldots, k_n) \in \mathcal{K}(n)$, define $F_{(k)}$ to be the discrete distribution that places a mass of k_i/n on the observation X_i, $i = 1, \ldots, n$. Then we have

$$\hat{\theta}_j = \sum_{i=1}^{j+1} \binom{j+1}{i} (-1)^{i+1} \sum_{l \in \mathcal{L}(n)} p_n^{(i-1)}(\mathbf{1}, l) |\mathcal{M}(l)|^{-1} \sum_{k \in \mathcal{M}(l)} t(F_{(k)}),$$

which follows from the general formula and is directly computable. The factor $|\mathcal{M}(l)|^{-1}$ reflects the equal weights shared by bootstrap resamples from the same set $\mathcal{M}(l)$. Our algorithm computes $\hat{\theta}_j$ direct from our equation.

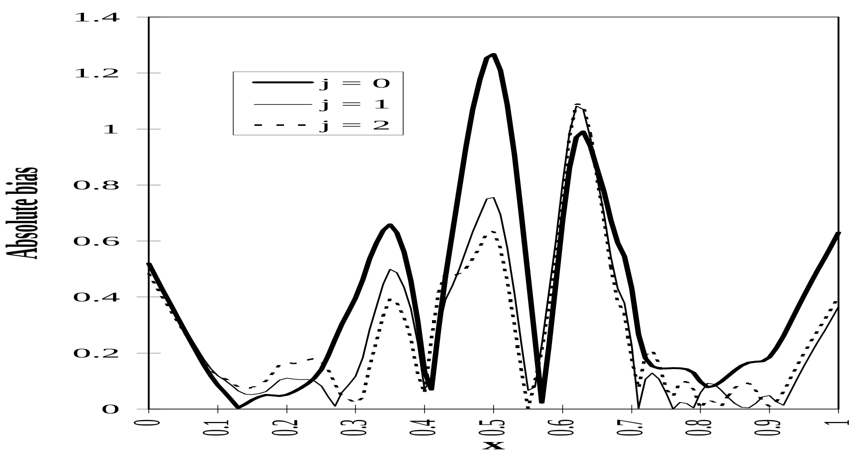
3. Simulation studies

Simulation studies are conducted to examine the effectiveness of our algorithm in reducing the bias of the Nadaraya-Watson estimate. Hall and Presnell (1999) apply their b-bootstrap algorithm to reduce the bias of the Nadaraya-Watson estimator. Our example is taken from their simulation study but focuses on a smaller sample size without fixed design points.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent data pairs such that X_i is uniformly distributed over $[0,1]$ and Y_i is normally distributed with mean $m(X_i)$ and variance $1/2$ conditional on X_i, where $m(x) = 5\{1 - x + \exp[-100(x-1/2)^2]\}$. Consider the Nadaraya-Watson estimator of the mean curve $m(x)$, defined as

$$\hat{m}(x) = \sum_{i=1}^{n} K_h(x - X_i)Y_i / \sum_{i=1}^{n} K_h(x - X_i),$$

where $K_h(u) = h^{-1}K(u/h)$, $K(u) = (2\pi)^{-1/2}\exp(-u^2/2)$ and the bandwidth h is set to 0.04. Denote by $\hat{m}_j$ the bias-corrected version of $\hat{m}$ based on j bootstrap iterations. The result is based on 50 random samples and sample size n is 10 throughout the study. The curves for $j \geq 2$ are visually indistinguishable. We see that bootstrap iteration is effective in reducing the bias of $\hat{m}(x)$, especially for moderate values of x. Convergence of the process is reached at approximately the second level and any further iterations do not improve the estimator significantly.



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