Some Remarks on the Frisch–Waugh Theorem

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In this paper we consider the partitioned linear regression model

\[ M = \{ y, (X_1 : X_2)\beta, V \}, \]

i.e.,

\[ y = X_1\beta_1 + X_2\beta_2 + \epsilon, \]

where \( X = (X_1 : X_2) \) is a given model matrix, \( V \) is a known nonnegative definite matrix, \( \beta = (\beta'_1 : \beta'_2)' \) with \( \beta_1 \) and \( \beta_2 \) being vectors of unknown parameters. Vector \( y \) is an observable random vector with expectation \( X\beta \) and dispersion matrix \( D(y) = V \). The unobservable vector \( \epsilon \) has expectation \( 0 \) and dispersion matrix \( D(\epsilon) = V \).

If the model \( M \) is premultiplied by \( M_1 = I_n - P_1 \) with \( P_1 \) being the orthogonal projector onto \( C(X_1) \), the column space of \( X_1 \), then we obtain the reduced model \( M_1 y = M_1 X_2 \beta_2 + M_1 \epsilon \), or shortly \( M_r = \{ M_1 y, M_1 X_2 \beta_2, M_1 VM_1 \} \). The so-called Frisch–Waugh Theorem states that (assuming \( X \) has a full column rank) the ordinary least squares estimator (OLSE) of \( \beta_2 \) under this reduced model equals the OLSE of \( \beta_2 \) under the original partitioned model. Davidson and MacKinnon (1993, p. 19) use the name Frisch–Waugh–Lovell Theorem, after Frisch and Waugh (1933) and Lovell (1963): ‘... since those papers seem to have introduced, and then reintroduced, it to econometricians.’

A generalized version of the Frisch–Waugh Theorem to the case of possibly singular \( V \) and possibly non-estimable \( \beta_2 \) claims that every BLUE of \( M_1 X_2 \beta_2 \) under the reduced model \( M_r \) remains BLUE of \( M_1 X_2 \beta_2 \) under the partitioned model \( M \), see e.g. Gross and Puntanen (1998a) and Bhimasankaram and Sengupta (1996, Theorem 6.1).

Since we are not interested in the parameter vector \( \beta_1 \), we may consider a reduction of the model \( M \) by a transformation of \( y \) into \( Fy \), where \( F \) is any matrix such that \( FX_1 = 0 \). Hence, we obtain the reduced model \( M_r(F) = \{ Fy, FX_2\beta_2, FVF' \} \) subject to \( FX_1 = 0 \).

We may now pose the following problem: For which transformation matrices \( F \) (where \( FX_1 = 0 \)) is every representation of the BLUE of \( M_1 X_2 \beta_2 \) in the reduced model \( M_r(F) \) also BLUE of \( M_1 X_2 \beta_2 \) in the partitioned model \( M \)?

The question above can be answered by employing the relationship between the concept of
linear sufficiency and the appropriate reduction of linear models. We may recall that a linear transformation $Fy$ (where not necessarily $FX_1 = 0$) is called linearly sufficient for an estimable vector of parametric functions $K\beta$ in model $M$ if and only if there exists a matrix $A$ such that $AFy$ is BLUE of $K\beta$. In that case, every representation of the BLUE of $K\beta$ in the induced model $M(F) = \{Fy, FX_1\beta_1 + FX_2\beta_2, FVF'\}$, is also BLUE in the partitioned model $M$, cf. Baksalary and Kala (1986).

As was mentioned earlier, the choice $F = M_1$ is valid to satisfy the required property that the BLUE of $M_1X_2\beta_2$ in the reduced model $M_r(M_1)$ is also BLUE in the partitioned model $M$. The model $M_r(M_1)$ has been studied for example in Puntanen (1997), Bhimasankaram et al. (1996, 1997) and Gross and Puntanen (1998a). The obtained results can be seen as generalizations of the Frisch–Waugh Theorem to the case of possibly singular $V$ and possibly non-estimable $\beta_2$.

The following result [for a proof, see Gross and Puntanen (1998b)] demonstrates that further choices for $F$ are possible:

*Let $F$ be any matrix satisfying the three conditions:

$$FX_2 = 0, \ N(F) \leftrightarrow C(M_1X_2) = \{0\}, \ C(FX_2) \leftrightarrow C[FV(X_1 : X_2)^\top] = \{0\}.$$  

Then every representation of the BLUE of $M_1X_2\beta_2$ in the reduced model $M_r(F)$ is also the BLUE of $M_1X_2\beta_2$ in the partitioned model $M$.  

**REFERENCES**


