APPLICATIONS OF BOOTSTRAP IN TWO-STAGE SHRINKAGE ESTIMATION

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1. Introduction


In this paper, we develop a new method for the estimation of a normal mean based on two-stage shrinkage method of Waikar, et al (1984) and the bootstrap method [Efron and Tibshirani (1993)]. Then, using the simulations, we show that the estimators of a normal mean obtained by this new method have better efficiency as compared to the efficiency of their known counterparts.

2. Two-stage Estimation for Normal Mean (Waikar Estimation)

Waikar et al. (1984) proposed the following [steps(1)-(4)] estimation scheme. Let \( X \sim N(\mu, \sigma^2) \) where \( \sigma^2 \) is known. Let \( \mu_0 \) be the initial estimate of \( \mu \) based on prior knowledge. Then; (1) Select two positive integers \( n_1 \) and \( n_2 \). (2) Obtain independent random observations \( X_{1i}, i = 1, 2, \ldots, n_1 \) (first stage sample). Let \( y_1 \) denote their mean. (3) Test \( H_0: \mu = \mu_0 \) versus \( H_1: \mu \neq \mu_0 \) at level \( \alpha \) using the first stage sample, and if \( H_0 \) is accepted the estimator of \( \mu \) is defined as

\[ \hat{\mu}_W = k \bar{X}_1 + (1 - k) \mu_0 \]

where \( k = \frac{|Z|}{Z_{(\alpha/2)}} \), \( Z = \frac{(\bar{X}_1 - \mu_0)}{\sigma \sqrt{n_1}} \), and \( Z_{(\alpha/2)} \) is the upper 100 \((\alpha/2)\) percentage point of a standard normal distribution. Here we note that if \( H_0 \) is accepted then \( k < 1 \) and the resulting estimator has the characteristic of shrinkage toward the prior mean \( \mu_0 \). (4) If \( H_0 \) is rejected, obtain a second sample of size \( n_2 \), \( X_{2i}, i = 1, 2, \ldots, n_2 \) and take the estimator of \( \mu \) as the pooled sample mean

\[ \hat{\mu}_P = (n_1 \bar{X}_1 + n_2 \bar{X}_2) / (n_1 + n_2) \]

The above estimator of \( \mu \) when compared with the usual estimator \( \bar{X} \) based on a fixed sample of an equivalent size \( n^* = E(n|\mu_0) = n_1 (1 + \alpha \mu) \) where \( u = n_2/n_1 \) was found to be more efficient.

For \( \sigma \) unknown, in step (3), the t-statistic is used for testing \( H_0 \) with related changes.

In view of improving the efficiency of the two-stage shrinkage estimators the choice of \( k \), as to be expected, plays an important role. In the above Waikar estimator the shrinkage factor \( k \) is fixed for a given first sample and a given value of \( \alpha \). In our new method due to bootstrapping factor \( k \) becomes a variable quantity and allows us to select the appropriate value of \( k \) for improving the efficiency of the shrinkage estimator.

3. Bootstrapping the Two-stage Shrinkage Estimator (Waikar – Ratnaparkhi)

In this methodology we use the bootstrap method for generating different values of \( k \) in the first stage ( i.e when we accept \( H_0 \)) of Waikar method as follows:

1. Generate a bootstrap sample of size \( n_1 \), \( X_{1*1}, X_{1*2}, \ldots, X_{1*m} \) say, from the first stage sample \( X_{11}, X_{12}, \ldots, X_{1m} \) of size \( n_1 \).
2. For each bootstrap sample we calculate \( k^* = |Z^*| / Z_{(\alpha/2)} \) where the '*' denotes the value of
3. If $k'<1$ then the bootstrap is considered as an acceptable bootstrap sample for this procedure. Otherwise it is discarded.

4. The above steps (1) – (3) are repeated until the required number of acceptable bootstrap samples, $b$ say, are generated.

5. The above steps (1) – (4) generate $b$ values of $k^*$, denoted by $k_1^*, k_2^*, ..., k_b^*$.

6. Calculate the mean $\overline{k}$ of these $b$ values of $k^*$.

Now the bootstrap estimator of $\mu$ denoted by $\hat{\mu}_{WR}$ is defined as:

$$\hat{\mu}_{WR} = \overline{k} \cdot \overline{X} + (1 - \overline{k})\mu_0$$

when $H_0$ is accepted, and

$$\hat{\mu}_{WR} = \left(\frac{n_1X_1 + n_2X_2}{n_1 + n_2}\right)$$

when $H_0$ is rejected.

The stage two procedure remains the same as in Waikar et al. (1984).

Next, using the simulations, we show that the new estimator $\hat{\mu}_{WR}$ is more efficient than $\hat{\mu}_W$ and $\hat{\mu}_B$ (where $\hat{\mu}_B$ is the regular bootstrap estimator of the mean). For the extended comparison of the MSE's of $\hat{\mu}_W$, $\hat{\mu}_{WR}$, and $\hat{\mu}_B$, a number of simulations for different sample sizes and different values of $\mu$, $\mu_0$ and $\alpha$ were carried. A few simulation results for $\sigma$ known (or $\sigma$ unknown) for $b=100$ bootstrap samples are given below.

**Table 1.**

<table>
<thead>
<tr>
<th>Simulation Results ($\sigma$ known)</th>
<th>Simulation Results ($\sigma$ unknown)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_1 = 50$ $n_2 = 70$ $Z_{(\alpha/2)} = 1.960$ $\overline{k} = 10$. $\mu = 10$. $\alpha = 0.05$</td>
<td>$n_1 = 20$ $n_2 = 20$ $T_{(\alpha/2)} = 2.093$ $\overline{k} = 10$. $\mu = 10$. $\alpha = 0.05$</td>
</tr>
<tr>
<td>$MSE \hat{\mu}_W$</td>
<td>$MSE \hat{\mu}_B$</td>
</tr>
<tr>
<td>1.57</td>
<td>4.10</td>
</tr>
<tr>
<td>1.59</td>
<td>4.09</td>
</tr>
<tr>
<td>1.58</td>
<td>4.06</td>
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</tbody>
</table>

**REFERENCES**


**RÉSUMÉ**

Dans cet article, en utilisant le bootstrapping, on développe une nouvelle méthode pour l'estimation à deux étages du rétrécissement de la moyenne normale. Le nouvel estimateur s'avère plus efficace que les estimateurs à deux étages connus auparavant.